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THE SCHWINGER VARIATIONAL METHOD FOR THREE BODY COLLISIONS

BY

SIDNEY BOROWITZ

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ERRATA

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Page 9 Line 3 read ..."particle 1" for ..."particle 2".

Page 9 Last paragraph until end of report, interchange Θ and Φ.

Page 10 Equation (3.15) read $\phi_n^*(r_2')$ for $\phi^*(r_2')$.

Page 11 Equation (3.16) read $\phi_n^*(r_1')$ for $\phi^*(r_1')$.

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ABSTRACT

The Schwinger variational method is formulated for three-body collisions. When exchange effects can be excluded, the generalization from two to three body problems is straightforward. Exchange effects, however, introduce difficulties in the latter problem which do not occur in the former. Despite this, the method can be extended to give a stationary expression for the exchange scattered amplitude. As a result of this formulation, one gains some insight into a superior method for treating three body problems with exchange in the Born approximation.

I. Introduction

A controlled experimental investigation of the physical processes in the ionosphere presents enormous difficulties inasmuch as the conditions which exist there cannot be reproduced terrestrially. The best one could hope for would be to isolate some physical processes to be studied on earth. In order to apply the results of such experiments to ionospheric data, it is important to know on which physical parameters the experimentally measured quantities depend, and what this dependence is. A considerable amount of information could be obtained from a theoretical investigation of these fundamental processes. If one finds that the results of terrestrial experiments can be predicted theoretically, one can with some assurance extrapolate to the conditions existing in the ionosphere. Thus theoretical investigations are of some importance both as a guide to experimental work and for the prediction of ionospheric effects.

The difficulty in the theoretical approach, however, lies in the fact that even the simplest problems involve many particles. Just as in classical mechanics, quantum theory does not, in general, permit any rigorous solution for this type of problem. A program for the theoretical investigation of the physical processes in the ionosphere must necessarily begin with a mathematical investigation of suitable approximation methods.

In this report such a program is begun in connection with scattering problems. This does not constitute by any means the first attempt for the solution of problems of this type. Mott and Massey[1] and their collaborators have solved many-particle collision problems of all sorts using principally the Born approximation. This method assumes that the incoming wave is distorted very little by the scattering center.

Because of the low temperature of the ionosphere the particles there possess energies of only a fraction of an electron-volt. Under these circumstances, the scattered particle spends so much time in the neighborhood of the scatterer that the approximation of small distortion of the incident particle is no longer valid.

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1. N. F. Mott and H.S.W. Massey, Theory of Atomic Collisions, Ch. 8, Oxford University Press.

An additional difficulty arising in three body collisions is in the possibility that if the incident particle is identical with one of the particles in the scatterer, exchange collisions may take place; i.e., particle 1 may come in and be captured and particle 2 emerge. According to the Pauli principle, this type of scattering cannot be distinguished from direct scattering in which the incident particle 1 is itself scattered. Exchange collisions may be of considerable importance for low energy encounters; and yet the theory of exchange collisions in any known approximation is not based on sound theoretical foundation. Calculations made according to existing procedures seem to be in serious disagreement with experiment[2].

Only recently have attempts been made to find other more suitable approximations. A variational method of Hulthén has been generalized to include three-body collisions[3]. Elastic and inelastic processes have also been treated extensively.[4,5,6]

The success of the application of the Schwinger variational procedure to two-body collisions[7,8] has made its generalization to three-body problems seem worthwhile. Another variational principle formulated in terms of Green's functions, while more difficult to apply, gives more accurate results. This report attempts to formulate such a variational principle which would be useful in discussing the scattering of electrons by hydrogen atoms both when exchange scattering is included and when it is not included. Some remarks will be made about the difficulties encountered when identical particles are involved and about the general usefulness of the Schwinger variational method in problems of this sort.

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2. Bates, Massey, Fundaminsky and Leech, Phil. Trans. of Roy. Soc. A,243,93 (1950)
3. W. Kohn, Phys. Rev. 74, 1793 (1948).
4. Su Shu Huang - Phys. Rev. 76, 477 (1949)
5. H. S. W. Massey and B. L. Moiseiwitsch, Proc. Roy. Soc. A205, 483 (1951).
6. M. Verde, Helv. Phys. Acta XXII, 339 (1949); A Troesch and M. Verde, Helv. Phys. Acta XXIV 39 (1951).
7. J. Schwinger, Lecture Notes on Nuclear Physics, Harvard University. J. Schwinger, B. A. Lippman, Phys. Rev. 79, 481 (1950).
8. Blatt and Jackson, Rev. of Mod. Phys. 22, 77 (1950).

II. Exchange Excluded

We consider in this section a collision of particle 1 incident on a bound system of particles 2 and 3. All three particles are different, so the complications arising from the Pauli principle can be neglected. The mass of particle 3 is taken to be infinite. This assumption is suitable for a discussion of the scattering of electrons by hydrogen atoms with exchange effects neglected. It would not be suitable for the description of the analogous nuclear problem, i.e., the scattering of neutrons by deuterons. For this latter problem the generalization of the work in Section II is straightforward, but the material in Section III could not be generalized so easily.

In all scattering problems the calculated quantity which is confronted with experiment is the differential scattering cross-section. This is defined as the current of scattered particles crossing unit area at infinity at some particular angle, divided by the current of particles per unit area in the incident beam. This number can be found by considering the solution of the Schrödinger equation, which is the following for this problem:

$$\left\{ -\frac{h^2}{2}\left[\frac{1}{m_1}\nabla_1^2 + \frac{1}{m_2}\nabla_2^2\right] + V_{13}(r_1) + V_{23}(r_2) + V_{12}(r_1,r_2) - E \right\}\psi(r_1,r_2) = 0. \tag{2.1}$$

In this equation r_1 and r_2 are the position vectors of particles 1 and 2 respectively. The position of particle 3 is taken as the origin of coordinates. The V's are the potentials between the particles, and we shall assume that the forces are central; E is the total energy of the system, i.e., the sum of the kinetic energy of the incident particle and the energy of the particle in its initial state (which is negative).

The asymptotic form of the solution of (2.1) should represent an incoming wave plus outgoing spherical waves. If we assume that particle 1 is incoming along the negative z axis, the asymptotic form of the solution is*

$$\underset{r_1\to\infty}{L}\,\psi(r_1,r_2) = e^{ik_o(\mu_z\cdot r_1)}\phi_o(r_2) + \sum_n f_n(\theta,\phi)\exp\frac{ik_n|r_1|}{|r_1|}\,\phi_n(r_2). \tag{2.2}$$

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*In what follows all summations include integrations over continuum states.

Here ϕ_n are the wave functions of the stationary state of the potential $V_{23}(r_2)$,

$$k_n^2 = \frac{2m_1}{\hbar^2}(E - E_n), \text{ and}$$

μ_z is a unit vector along the z axis, the direction of the incoming electron.

E_n is the energy of the n^{th} state of system 2 and 3.

If the asymptotic form of the solution is given by (2.1), the scattering cross-section for excitation of the bound system to the n^{th} state can be shown[1] to be

$$\sigma_n = \frac{k_n}{k_o} |f_n|^2 . \tag{2.3}$$

In order to derive a Schwinger variational method, it is necessary to replace equation (2.1) by an integral equation which satisfies the boundary condition (2.2). To achieve this result we transpose $V_{13}(r_1)$ and $V_{12}(r_1,r_2)$ in (2.1) to get

$$\left\{ \frac{\hbar^2}{2}\left[\frac{1}{m_1}\nabla_1^2 + \frac{1}{m_2}\nabla_2^2\right] + (E - V_{23}) \right\}\psi = (V_{13} + V_{12})\,\psi = V\psi , \tag{2.4}$$

where we have written V for $(V_{13} + V_{12})$.

The integral equation replacing (2.1) and satisfying (2.2) is given by

$$\psi^{\circ}_{\mu_z}(r_1,r_2) = \exp ik_o(\mu_z \cdot r_1)\,\phi_o(r_2) + \int dr_1'\,dr_2'\,G(r_1,r_2,\ r_1',r_2')\times v(r_1',r_2')\psi^{\circ}_{\mu_z}(r_1'r_2'). \tag{2.5}$$

In equation (2.5)

$$v(r_1', r_2') = \frac{2m_1}{\hbar^2} V(r_1', r_2') .$$

$\psi^{\circ}_{\mu_z}$ is a solution of (2.1) which asymptotically represents an incoming plane wave of particle 1 with energy k_o^2 and a bound state of particle 2 with energy E_o.

$G(r_1,r_2;\ r_1',\ r_2')$ is the Green's function of the operator

$$\left\{ \frac{\hbar}{2}\left[\frac{1}{m_1}\nabla_1^2 + \frac{1}{m_2}\nabla_2^2\right] + E - V_{23} \right\},$$ and satisfies the differential equation

$$\left\{ \frac{\hbar}{2}\left[\frac{1}{m_1}\nabla_1^2 + \frac{1}{m_2}\nabla_2^2\right] + E - V_{23} \right\} G(r_1,r_2;r_1',r_2') = +\delta(r_1-r_1')\,\delta(r_2-r_2'). \tag{2.6}$$

The Green's function can be found by the method of separation of variables to be explicitly

$$\begin{aligned} G &= -\sum_n \phi_n^*(r_2')\,\phi_n(r_2)\,\frac{\exp ik_n|r_1-r_1'|}{4\pi|r_1-r_1'|} \\ &= -\sum_n \phi_n^*(r_2)\,\phi_n(r_2')\,\frac{\exp ik_n|r_1-r_1'|}{4\pi|r_1-r_1'|} \quad . \end{aligned} \tag{2.7}$$

This result follows from the fact that

$$\sum \phi_n^*(r_2)\,\phi_n(r_2') = \delta(r_2-r_2'), \tag{2.8}$$

and that $\frac{\exp ik_n|r_1-r_1'|}{4\pi|r_1-r_1'|}$ is the Green's function for the operator $(\nabla_2^2 + k_n^2)$.

In the limit $r_1 \to \infty$, (2.7) becomes

$$\underset{r_1\to\infty}{L}\; G = -\sum_n \phi_n(r_2)\,\frac{\exp ik_n|r_1|^*}{|r_1|}\cdot\frac{1}{4\pi}\exp - ik_n(\mu\cdot r_1')\,\phi_n^*(r_2') . \tag{2.9}$$

Inserting (2.9) into (2.5) and comparing with (2.2) we have

$$4\pi\, f_n^o = -\int \exp\text{-}ik_n\,(\underline{\mu}\cdot \underline{r}_1')\,\phi_n^*(r_2')\,v(r_1'\,r_2')\Psi_{\mu_z}^o(r_1',r_2')\,dr_1'\,dr_2' , \tag{2.10}$$

for the scattered amplitude. The cross sections can be determined from (2.10) according to equation (2.3).

In order to formulate a variational principle for f_n^o , it is necessary to write down an integral equation adjoint to (2.5) which satisfies (2.4). This one is

$$\Psi_\mu^n(r_1,r_2) = \exp ik_n(\mu\cdot r_1)\,\phi_n(r_2) + \int G^*(r_1,r_2)(r_1',r_2')v(r_1',r_2')\Psi_\mu^n(r_1',r_2')\,dr_1'\,dr_2'. \tag{2.11}$$

The asymptotic form of this solution consists of concentrating waves. By multiplying the complex conjugate of (2.11) by $v\psi^{o}_{\mu_z}$ and integrating, one sees that

$$4\pi f^{o}_{n} = \int \psi^{*n}_{\mu}(r_1', r_2')\, v(r_1' r_2')\, \psi^{o}_{\mu_z}(r_1', r_2')\, dr_1'\, dr_2' \tag{2.12}$$

$$- \iint \psi^{*n}_{\mu}(r_1', r_2')\, v(r_1' r_2')\, G(r_1', r_2' ; r_1'' r_2'')\, v(r_1'', r_2'')\, \psi^{o}_{\mu_z}(r_1'', r_2'')\, dr_1' dr_2' dr_1'' dr_2'' .$$

By treating Equation (2.5) in an analogous way we have

$$4\pi f^{n}_{o} = \int \exp ik_o(\mu_z \cdot r_1)\, \phi_o(r_2')\, v(r_1', r_2')\, \psi^{*n}_{\mu}(r_1', r_2')\, dr_1'\, dr_2' = 4\pi f^{o}_{n}. \tag{2.13}$$

With the reciprocity relation (2.13), one can form the expression[9]

$$\frac{1}{4\pi f^{o}_{n}} = \frac{\displaystyle \int \psi^{*n}_{\mu}(r_1', r_2')\, v(r_1', r_2')\, \psi^{o}_{\mu_z}(r_1', r_2')\, dr_1', dr_2' - \iint \psi^{*n}_{\mu}(r_1', r_2')\, v(r_1', r_2')\, G(r_1', r_2'; r_1'', r_2'')\, v(r_1'', r_2'')\, \psi^{o}_{\mu_z}(r_1'', r_2'')\, dr_1' dr_2'\, dr_1'' dr_2'' .}{\displaystyle \int \exp ik_o(\mu_z \cdot r_1')\, \phi_o(r_2')\, v(r_1', r_2')\, \psi^{*n}_{\mu}(r_1', r_2')\, dr_1'\, dr_2' \int \exp -ik_n(\mu \cdot r_1')\, \phi^{*}_{n}(r_2')\, \psi^{o}_{\mu_z}(r_1', r_2')\, dr_1' dr_2'} \tag{2.14}$$

which is stationary with respect to arbitrary variations of $\psi^{o}_{\mu_z}$ and ψ^{*n}_{μ}. Thus if trial functions were substituted for $\psi^{o}_{\mu_z}$ and ψ^{*n}_{μ} on the right hand side of (2.14) which differed from the true solution by ε, the scattered amplitude f^{o}_{n}, would be correct to order ε^2, obtained from (2.14).

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9. N. Marcuvitz - Sec. III. D. "Recent Developments in the Theory of Wave Propagation", New York University Institute for Mathematics and Mechanics, 1949.

III. Exchange Included

If particles 1 and 2 are identical the results of Section II are not complete. Under these circumstances we have, as indicated in the Introduction, the possibility of exchange scattering, namely the possibility that particle 2 and 1 will change roles and the former will emerge when the latter is incident.

This exchange scattering will now affect the value for the total cross-section for the following reason. The Pauli principle imposes certain symmetry conditions on the wave functions. The complete solution (space and spin parts) must be antisymmetrical for Fermi particles and symmetrical for Bose particles. Since the applications we envisage are concerned with electron scattering, we shall restrict ourselves to the former case, in which the space part must be anti-symmetrical if the spins are parallel, and the space part must be symmetrical if the spins are anti-parallel. It can be shown[10] that the quantities which determine the total differential scattering cross-section are

$$s_n = |f_n + g_n|^2$$

and (3.1)

$$a_n = |f_n - g_n|^2 \; .$$

f_n^o is the ordinary scattering cross-section as determined in Section II, and

g_n^o is the exchange scattering cross-section which we have discussed above.

The total scattering cross-section is then given by a suitable weighted mean of the quantities in (3.1), namely

$$\sigma_n = \frac{k_n}{k_o} \left(\frac{1}{4} \; s_n + \frac{3}{4} \; a_n \right) . \qquad (3.2)$$

It is possible, therefore, to satisfy the Pauli exclusion principle without explicitly symmetrizing the result. This requires, however, that we know of the exchange scattering amplitude, g_n^o .

10. Mott and Massey, loc. cit. Chap. 8.

The formulation of the problem given in Section II is not convenient for describing the possibility that particles 1 and 2 exchange their roles. In addition, Kohn[11] has pointed out that it is not certain whether any iterative procedure based on equation (2.5) could describe a bound state in particle 1 and a continuum state in particle 2. Both difficulties can be circumvented by treating both particles in a symmetric way. We rewrite equation (2.1)

$$\left\{\nabla_1^2 + \nabla_2^2 + e - v_{13}(r_1) - v_{23}(r_2)\right\}\psi = v\,(r_1,r_2)\,\psi \tag{3.3}$$

where

$$e = \frac{2m_1}{\hbar^2}\,E$$

$$v(r_1,r_2) = \frac{2m}{\hbar^2}\,V_{12}$$

$$v_{13} = \frac{2m_1}{\hbar^2}\,V_{13}$$

$$v_{23} = \frac{2m_1}{\hbar^2}\,V_{23} \quad .$$

The integral equation corresponding to (3.3) is

$$\psi = \phi_i(r_1)\phi_o(r_2) + \int G\left(r_1,r_2;r_1',r_2'\right)v_{12}(r_1',r_2')\,\psi(r_1',r_2')\;dr_1'\;dr_2' \tag{3.4}$$

where $G(r_1,r_2;r_1',r_2')$ is the Green's function of the operator

$$L = \nabla_1^2 + \nabla_2^2 + e - v_{13} - v_{23} \tag{3.5}$$

and $\phi_i(r)$, $\phi_o(r)$ are solutions of the equations

$$\begin{aligned}(\nabla^2 + e_o - v_{23})\,\phi_o &= 0\\ (\nabla^2 + (e-e_o) - v_{13})\,\phi_i &= 0;\end{aligned} \tag{3.6}$$

e_o = ground state energy of the system consisting of particles 2 and 3.

11. W. Kohn - private communication.

In order that the second part of (3.4) give the scattering cross-section directly, ϕ_i at infinity should represent a wave coming in along the direction of particle 2, with energy $k_o^2 = e-e_o$.

The Green's function has two representations which again can easily be found by separation of variables using the relation (2.8):

$$G(r_1,r_2;\, r_1',r_2') = -\sum_n \phi_n(r_1)\phi_n^*(r_1')\; g_{k_n}(r_2,r_2') \tag{3.7a}$$

$$= -\sum \phi_n(r_2)\,\phi_n^*(r_2')\; g_{k_n}(r_1,r_1')\,. \tag{3.7b}$$

If v_{12} falls off faster than $\frac{1}{r_1}$ (or $\frac{1}{r_2}$)*, the asymptotic form of (3.7a) with $r_2 \to \infty$ is[12]

$$\underset{r_2\to\infty}{\mathrm{LG}} = -\sum_n \phi_n(r_1)\,\phi_n^*(r_1')\;\frac{e^{ik_n|r_2|}}{|r_2|}\;\frac{1}{4\pi}\;F_{k_n}(r_2',\pi-\Theta), \tag{3.8a}$$

and the limit as $r_1 \to \infty$ can be found from (3.7b):

$$\underset{r_1\to\infty}{\mathrm{LG}} = -\sum \phi_n(r_2)\,\phi_n^*(r_2')\;\frac{e^{ik_n|r_1|}}{|r_1|}\;\frac{1}{4\pi}\;F_{k_n}(r_1',\pi-\Phi)\,. \tag{3.8b}$$

In (3.8a) and (3.8b) F_{k_n} represents the solution of the homogeneous equation (3.6) which is an incoming wave at infinity in the direction $\pi - \Theta$ or $(\pi - \Phi)$ with energy $e-e_n$. Θ and Φ are the angles between r_2 and r_2', and r_1 and r_1'

Using equations (3.8a) and (3.8b), we can now find an expression for f_n^o and g_n^o from (3.4):

$$\underset{r_1\to\infty}{\mathrm{Lim}}\,\psi \longrightarrow \phi_i(r_1)\phi_o(r_2) + \sum_n \phi_n(r_1)\,\frac{e^{ik_n|r_2|}}{|r_2|}\int -\frac{1}{4\pi}\,\phi_n^*(r_1')F_{k_n}(r_2',\pi-\Theta)v_{12}(r_1',r_2')\,\psi(r_1',r_2')dr_1'\,dr_2' \tag{3.9}$$

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*The Coulomb case will be discussed in Sec. IV.

12. Mott and Massey, loc.cit. Chap.4.

and

$$\lim_{r_2\to\infty} \longrightarrow \sum_n \phi_n(r_2) \frac{e^{ik_n|r_1|}}{|r_1|} \int -\frac{1}{4\pi} \phi_n^*(r_2')F_{k_n}(r_1',\pi-\Phi)v_{12}(r_1',r_2') \psi(r_1',r_2')\, dr_1'\, dr_2' \; ; \tag{3.10}$$

thus

$$f_n^o = -\frac{1}{4\pi}\int \phi^*(r_1')F_{k_n}(r_2',\pi-\Theta)v_{12}(r_1',r_2')\psi(r_1',r_2')\, dr_1'\, dr_2' \tag{3.11}$$

and

$$g_n^o = -\frac{1}{4\pi}\int \phi^*(r_2')F_{k_n}(r_1'\,\pi-\Phi)\, v_{12}(r_1',r_2')\psi(r_1',r_2')\, dr_1'\, dr_2' \,. \tag{3.12}$$

It should be noted here that the elastic scattered amplitude is not given completely by f_n^o calculated according to (3.11). The outgoing spherical part of $\phi_i(r_1)\, \phi_o(r_2)$ should be added to this value.

As in Section II, we can now write down stationary expressions for $1/4\pi f_n^o$ and $1/4\pi g_n^o$. We note that the equation adjoint to (3.4) for the calculation of f_n^o is

$$\chi(r_1,r_2) = F_{k_n}^*(r_2,\pi-\Theta)\phi_n(r_1) + \int G^*(r_1,r_2;r_1',r_2')\, v_{12}(r_1',r_2')\chi(r_1',r_2')\, dr_1'\, dr_2' \,. \tag{3.13}$$

The adjoint equation for the calculation of g_n^o is

$$\lambda(r_1,r_2) = F_{k_n}^*(r_1,\pi-\Phi)\phi_n(r_2) + \int G^*(r_1,r_2;r_1',r_2')v_{12}(r_1',r_2')\, \lambda(r_1',r_2')\, dr_1'\, dr_2' \,, \tag{3.14}$$

The reciprocity relations corresponding to (2.13) can again be proved in the same way, and we have for f_n^o and g_n^o following expressions which are stationary with respect to arbitrary variations of ψ, λ, and χ:

$$\frac{1}{4\pi f_n^o} = \frac{\int \chi^*(r_1',r_2')v_{12}(r_1',r_2')\psi(r_1',r_2')dr_1'dr_2' - \iint \chi^*(r_1',r_2')v_{12}(r_1',r_2')G(r_1',r_2';r_1'',r_2'') \times v_{12}(r_1'',r_2'')\psi(r_1'',r_2'')\, dr_1'\, dr_2'\, dr_1'', dr_2''}{\int \phi_i(r_1')\phi_o(r_2')v_{12}(r_1',r_2')\chi^*(r_1',r_2')dr_1'\, dr_2' \int F_{k_n}(r_1',\pi-\Theta)\phi^*(r_2')v_{12}(r_1',r_2')\psi(r_1',r_2')\, dr_1'\, dr_2'} \tag{3.15}$$

and
$$\frac{1}{4\pi g_n^o} = \frac{\int \lambda^*(r_1',r_2')v_{12}(r_1',r_2')\psi(r_1',r_2') - \iint \lambda^*(r_1',r_2')v_{12}(r_1',r_2')G(r_1',r_2';r_1'',r_2'') \times v_{12}(r_1'',r_2'')\psi(r_1'',r_2'')\,dr_1'dr_2'\,dr_1''\,dr_2''}{\int \phi_1(r_1')\phi_o(r_2')\,v_{12}(r_1',r_2')\lambda^*(r_1',r_2')dr_1'dr_2' \int F_{k_n}(r_2',\pi-\phi)\phi^*(r_1')v_{12}(r_1'r_2')\,\psi(r_1',r_2')\,dr_1'\,dr_2'} \quad (3.16)$$

Having evaluated f_n^o and g_n^o we can find the cross-section, including exchange, for spin $\frac{1}{2}$ particles by reference to (3.1) and (3.2).

IV. The Coulomb Case.

If the potentials v_{13} and v_{23} go to zero as $\frac{1}{|r_1|}$ and $\frac{1}{|r_2|}$ respectively, as would be the case if we were discussing the scattering of electrons by hydrogen, then some of the statements made in the preceding section are not precisely correct. We have the usual difficulty that the outgoing is not $\frac{e^{ik|r_1|}}{|r_1|}$ but $\frac{e^{ik|r_1|-i\log k|r_1|}}{|r_1|}$, and there are similar difficulties with the incoming wave.[13] We may carry through exactly the same program as we did in Section III after investigating the asymptotic form of the hydrogenic Green's function and after the correct interpretation of the scattering cross-section has been made for this case. The formulae derived are the same, provided we choose the ϕ_1 and F functions to represent the incoming waves corresponding to the Coulomb case. While no formal difficulties exist, the question of the convergence of an iterative procedure for the Coulomb field is still unsettled. This matter is rather complicated and will not be discussed in this report.

V. Evaluation of the Results

Having carried through the formulation of the Schwinger variational method for three body collisions, it is fair to ask what are the prospects for utilizing it in the computation of cross-sections which are of interest.

13. Mott and Massey loc.cit. p. 48.

At present it seems that it is necessary to treat the two particles in a symmetrical way; however, this requires the use of a Green's function so complicated that the difficulties involved in its use would be almost prohibitive. There are, however, some encouraging aspects. If the conjecture of Kohn[11] is correct, it may be worthwhile to examine the symmetrical treatment in the Born approximation. This may remove some of the previous difficulties[2] encountered in the calculations including exchange. As a final heartening note, if one excludes exchange effects and follows the formulation of Section II, the integrals involved are tractable[14]. In fact, using a Born trial field, the integrals can be explicitly evaluated except for some minor approximations.

VI. Acknowledgments

I have profited from discussions with Professor B. Friedman, E. Gerjuoy, W. Kohn and N. Marcuvitz.

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14. An application of this sort will be the subject of a future report by H. Boyet and S. Borowitz.